

CHARACTERISTIC, MAXIMUM MODULUS AND VALUE DISTRIBUTION

BY

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ABSTRACT. Let f be an entire function such that $\log M(r, f) \sim T(r, f)$ on a set E of positive upper density. Then f has no finite deficient values. In fact, if we assume that E has density one and f has nonzero order, then the roots of all equations $f(z) = a$ are equidistributed in angles. In view of a recent result of Murai [6] the conclusions hold in particular for entire functions with Fejér gaps.

1. Introduction. In a recent paper Murai [6] proved among other things that if $f(z) = \sum_0^\infty a_n z^{\lambda_n}$ is an entire function with Fejér gaps, i.e.

$$(1.1) \quad \sum \lambda_n^{-1} < \infty,$$

then $f(z)$ can have no deficient values. In the course of his proof Murai showed that for such a function

$$(1.2) \quad T(r, f) \sim \log M(r, f)$$

as $r \rightarrow \infty$ outside a set of finite logarithmic measure, where $T(r, f)$ is the Nevanlinna characteristic and $M(r, f)$ the maximum modulus of f . In this paper we show that the condition (1.2) suffices in order that a transcendental entire function should have no deficient values and, subject to certain growth conditions, that the roots of all equations $f(z) = a$ are equidistributed in angles. It is clear that some additional growth condition is necessary for this. In fact if $f(z)$ is an entire function of genus zero, $n(r)$ is the counting function of its zeros and

$$N(r) = \int_0^r \frac{n(t) dt}{t},$$

then [4, (4.11), p. 133]

$$(1.3) \quad n(r) = o\{N(r)\}$$

implies (1.2), but (1.3) is unaffected by the arguments of the zeros. We shall see that a weaker gap condition than (1.1), namely Fabry gaps

$$(1.4) \quad \lambda_n/n \rightarrow \infty,$$

is sufficient or alternatively a growth condition, namely that $f(z)$ has positive order and satisfies (1.2) on a set of density one.

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2. Statement of results. We take for granted the usual notation of Nevanlinna theory. Let $f(z)$ be a transcendental entire function of order λ and lower order μ , where $0 \leq \mu \leq \lambda \leq \infty$.

THEOREM 1. *Suppose that $f(z)$ is an entire function such that*

$$(2.1) \quad \lim_{r \rightarrow \infty} \frac{T(r, f)}{\log M(r, f)} = 1$$

as $r \rightarrow \infty$ on a set E of positive upper density δ . Then there exists a set F of density zero, such that for every complex a we have

$$(2.2) \quad N(r, a, f) \sim T(r, f)$$

as $r \rightarrow \infty$ in $E \setminus F$. In particular, $\delta(a, f) = 0$ for every a .

We write $n(r, \theta_1, \theta_2, a)$ for the number of roots of the equation $f(z) = a$ in the sector

$$S(r, \theta_1, \theta_2): 0 < |z| < r, \quad \theta_1 < \arg z < \theta_2,$$

and

$$(2.3) \quad N(r, \theta_1, \theta_2, a) = \int_0^r \frac{n(t, \theta_1, \theta_2, a) dt}{t}.$$

Our next result is

THEOREM 2. *If $\lambda > 0$ and $f(z)$ satisfies (2.1) as $r \rightarrow \infty$ on a set E_1 of density one, then there exists a set E_2 of upper density one such that*

$$N(r, \theta_1, \theta_2, a) \sim \frac{\theta_2 - \theta_1}{2\pi} T(r, f)$$

as $r \rightarrow \infty$ on E_2 for every complex a and every pair θ_1, θ_2 such that $\theta_1 < \theta_2 \leq \theta_1 + 2\pi$.

Our results have a natural extension to subharmonic functions when we consider the Riesz mass on set G of a subharmonic function $u(z)$ to be the analogue of the number of zeros on G of the function $f(z) - a$. We can then apply the subharmonic result to $u(z) = \log|f(z) - a|$, provided that the set G is chosen independent of a .

3. A growth result for real functions. In order to obtain Theorems 1 and 2 we prove an extension of a growth lemma of Edrei and Fuchs [1] to entire functions of arbitrary growth. Such an extension is possible if we work with the maximum modulus instead of the characteristic. However, in order to do this we need a sharpened version of an inequality for real functions of Hayman and Stewart [5]. We assume in this section that $f(x)$ is a real function such that for sufficiently large positive x , $f^{(n-1)}(x)$ is convex. Thus for large x , $f^{(n)}(x)$ exists and is increasing outside a countable set. If, in addition, $f^{(n)}(x) > 0$ for large x , we say that $f(x) \in B(n)$ and define

$$f_n(x) = \inf_{h>0} \frac{f(x+h)}{h^n}.$$

It was proved in [5] that for $f(x) \in B(n)$ we have, given $K > 1$,

$$(3.1) \quad f_n(x) < (eK/n)^n f^{(n)}(x)$$

on a set E of positive lower density. In this paper we need to prove that the lower density is close to one if K is large. More precisely we have

THEOREM 3. *If E is the set of all x for which (3.1) is true when $f(x) \in B(n)$, then if $\delta(E)$ denotes the lower density of E , we have $\delta(E) \geq (K - 1)/(K - 1 + n)$.*

We follow the argument of [5] and define

$$\beta(x) = \sup_{0 \leq \nu \leq n-1} \left\{ \frac{f^{(\nu)}(x)}{f^{(n)}(x)} \right\}^{1/(n-\nu)}.$$

We need

LEMMA 1. *If $f(x) \in B(n)$ and $f, f', \dots, f^{(n)}$ are all positive for $x \geq x_0$, then $x - \beta(x)$ is increasing for $x \geq x_0$.*

In [5, Lemma 3] it was shown that

$$(3.2) \quad \beta\{x + \delta\beta(x)\} \leq e^\delta \beta(x).$$

Suppose that there exist x_1 and x_2 , such that $x_0 \leq x_1 < x_2$ and $x_2 - \beta(x_2) < x_1 - \beta(x_1)$. Then there exists $C > 1$ such that $Cx_2 - \beta(x_2) = Cx_1 - \beta(x_1)$. We write for a large positive integer N

$$h = (x_2 - x_1)/N, \quad \xi_j = x_1 + jh, \quad j = 0, \dots, N,$$

and deduce that for at least one j , $0 \leq j \leq N - 1$, we have

$$C\xi_{j+1} - \beta(\xi_{j+1}) \leq C\xi_j - \beta(\xi_j),$$

i.e.

$$\beta(\xi_j + h) \geq \beta(\xi_j) + Ch = \beta(\xi_j)\{1 + Ch/\beta(\xi_j)\}.$$

Writing $h = \delta\beta(\xi_j)$ we obtain

$$\beta(\xi_j + h) \geq \beta(\xi_j)(1 + C\delta) > e^\delta \beta(\xi_j)$$

if δ is sufficiently small, i.e. N sufficiently large, since $C > 1$. This contradicts (3.2) and proves Lemma 1. We deduce

LEMMA 2. *Suppose that $0 < \theta < 1$ and $C > 0$. Then for x on a set of lower density at least $(1 - \theta)/(1 - \theta + \theta C)$ we have*

$$(3.3) \quad \beta(x + h) > \theta\beta(x) \quad \text{for } 0 \leq h \leq C\theta\beta(x).$$

We note that $\beta(x)$ is continuous except on the countable set of jump increases of $f^{(n)}(x)$, where $\beta(x)$ has a jump decrease. At these points we define $\beta(x) = \beta(x + 0)$, so that $\beta(x)$ is continuous to the right. We suppose $x_0 = x'_0$ to be as in Lemma 1, and if x'_{j-1} has already been defined, we define x_j to be the lower bound and so the least value of $x \geq x'_{j-1}$ such that

$$\beta(x + h) \leq \theta\beta(x) \quad \text{for some } h \leq C\theta\beta(x).$$

We then choose the least such h and set $x'_j = x_j + h$. Let E be the set of all x in the union of the intervals (x'_j, x_{j+1}) . Then it is evident that (3.3) holds in E . It remains to estimate the lower density of E .

Suppose then that $X > x_0$, and assume first that $X = x'_p$ for some $p > 0$. Since $x - \beta(x)$ is nondecreasing we note that

$$\sum_{j=0}^p \{x'_j - x_j - \beta(x'_j) + \beta(x_j)\} \leq X - x_0 - \beta(X) + \beta(x_0) < X + O(1).$$

Again by our construction

$$\beta(x_j) - \beta(x'_j) \geq (1 - \theta)\beta(x_j) \geq \frac{(1 - \theta)}{\theta C}(x'_j - x_j).$$

Thus

$$(3.4) \quad \left\{1 + \frac{1 - \theta}{\theta C}\right\} \sum_{j=0}^p (x'_j - x_j) < X + O(1).$$

So if $E(X) = E \cap [x_0, X]$ and $|E(X)|$ denotes the length of $E(X)$ we see that

$$(3.5) \quad |E(X)| \geq X \left\{1 - \frac{C\theta}{C\theta + (1 - \theta)}\right\} + O(1) = \frac{(1 - \theta)}{C\theta + 1 - \theta} X + O(1).$$

Next if $x_p \leq X \leq x'_p$, X is smaller while $|E(X)|$ is the same, so that (3.5) is still valid. Again if $x'_p \leq X < x_{p+1}$, X is larger, so that (3.4) and (3.5) are still valid. Thus (3.5) holds in all cases and Lemma 2 is proved.

LEMMA 3. Suppose that for some numbers $x = x_0$, θ and C we have (3.3). Then

$$(3.6) \quad f\{x_0 + C\theta\beta(x_0)\} \leq \{\beta(x_0)\}^n e^C f^{(n)}(x_0).$$

We write

$$\beta = \theta\beta(x_0), \quad \alpha = (\beta/\theta)^n f^{(n)}(x_0), \quad \varphi(x) = \alpha \exp\{(x - x_0)/\beta\},$$

and suppose that (3.6) is false. From this we shall obtain a contradiction to (3.3).

We define

$$x_2 = \inf\{x, x_0 \leq x \text{ and for some } \nu, 0 \leq \nu \leq n, f^{(\nu)}(x) > \varphi^{(\nu)}(x)\}.$$

Since (3.6) is false we have

$$\varphi(x_0 + C\beta) = \alpha e^C = (\beta/\theta)^n e^C f^{(n)}(x_0) < f(x_0 + C\beta).$$

Again for $\nu = 0, \dots, n$ we have

$$\varphi^{(\nu)}(x_0) = \alpha/\beta^\nu = \theta^{-\nu} \beta^{n-\nu} f^{(n)}(x_0) \geq \beta(x_0)^{n-\nu} f^{(n)}(x_0) \geq f^{(\nu)}(x_0).$$

Thus x_2 exists and $x_0 \leq x_2 < x_0 + C\beta$.

Suppose now that for some $\nu < n$ we have

$$(3.7) \quad \varphi^{(\nu)}(x_2) \leq f^{(\nu)}(x_2).$$

Then we have by the definition of x_2

$$(3.8) \quad \varphi^{(\nu)}(x) \geq f^{(\nu)}(x), \quad 0 \leq x < x_2.$$

Hence we deduce that

$$\frac{d}{dx} \frac{\varphi^{(\nu)}(x)}{f^{(\nu)}(x)} \leq 0$$

at $x = x_2$, where differentiation denotes the left derivative. Thus

$$1 \leq \frac{f^{(\nu)}(x_2)}{\varphi^{(\nu)}(x_2)} \leq \frac{f^{(\nu+1)}(x_2)}{\varphi^{(\nu+1)}(x_2)},$$

so that (3.7) holds with $\nu + 1$ instead of ν . Thus finally (3.7) must hold with $\nu = n$ for the left derivative and so also the right derivative, while by the definition of x_2 we have (3.8) for $\nu < n$ and $x < x_2$ and by continuity also for $x = x_2$. Thus

$$\frac{f^{(\nu)}(x_2)}{f^{(n)}(x_2)} \leq \frac{\varphi^{(\nu)}(x_2)}{\varphi^{(n)}(x_2)} = \beta^{n-\nu}, \quad \nu = 0, \dots, n-1,$$

so that $\beta(x_2) \leq \beta = \theta\beta(x_0)$. This contradicts (3.3) and so Lemma 3 is proved.

We can now complete the proof of Theorem 3. We set $h = C\theta\beta(x_0)$ and deduce from (3.6) that if (3.3) holds with $x = x_0$ then

$$f_n(x_0) \leq \frac{f(x_0 + h)}{h^n} \leq \frac{e^C}{(C\theta)^n} f^{(n)}(x_0).$$

By Lemma 2 we deduce that this inequality holds in a set of lower density at least $\delta = (1 - \theta)/(1 - \theta + \theta C)$. Setting $C = n$, $\theta = K^{-1}$ we deduce Theorem 3.

4. Proof of Theorem 1. In this section we suppose that $u(z)$ is subharmonic and not constant in the plane and that $u(0) = 0$. We write

$$(4.1) \quad B(r) = \sup_{|z|=r} u(z),$$

$$(4.2) \quad b(r) = \int_0^r (r-t)B(t) dt, \quad b_2(r) = \inf_{h>0} \frac{b(r+h)}{h^2},$$

so that $b''(r) = B(r)$. We also write $n(z, h)$ for the Riesz mass of u in the disk $|\xi - z| \leq h$ and set

$$(4.3) \quad N(z, h) = \int_0^h \frac{n(z, t)}{t} dt,$$

$$(4.4) \quad u(z, h) = u(z) + N(z, h) = \frac{1}{2\pi} \int_0^{2\pi} u(z + he^{i\theta}) d\theta.$$

Suppose that $f(z)$ is a transcendental entire function and that a is a complex constant. Then we have

$$(4.5) \quad f(z) - a = c_\lambda z^\lambda + \dots$$

and will apply our results to

$$(4.6) \quad u_a(z) = \log \left| \frac{(f(z) - a)}{c_\lambda z^\lambda} \right| = \log |f_a(z)|.$$

We denote by $A_1, A_2, A_3, \dots$ positive absolute constants. We need

LEMMA 4. If $0 < |z| = r < R$ and $h = A_1(R - r)$, $0 < A < 1$, we have

$$(4.7) \quad u\left(z, \frac{1}{2}h\right) > -\frac{A_2}{(R - r)^2}b(R)$$

and

$$(4.8) \quad n(z, h) < \frac{A_3}{(R - r)^2}b(R).$$

Further if $0 < d < \frac{1}{2}h$ we have

$$(4.9) \quad N\left(\zeta, \frac{1}{2}h\right) < \frac{A_4}{(R - r)^2} \log\left(\frac{16h}{d}\right)b(R)$$

for $|\zeta - z| < \frac{1}{2}h$ except possibly when ζ lies in a set of disks, the sum of whose radii is at most d .

The conclusions (4.7)–(4.9) are (14.1)–(14.3) of [2, p. 494]. The quantity $b(r)$ of the present paper is the $B_2(r)$ of [2].

We now prove

THEOREM 4. With the above notation there exists an absolute constant A_5 , such that if $K > 0$, we have

$$(4.10) \quad u(re^{i\theta}) > -Kb_2(r)$$

for $0 \leq \theta \leq 2\pi$, outside a set $e(r, K)$ of θ whose measure is at most $4\pi \exp(-A_5K)$.

We start by finding R , such that $r < R \leq 2r$ and

$$(4.11) \quad b(R)/(R - r)^2 \leq 4b_2(r).$$

If $R > 2r$, we deduce from the fact that $B(r)$ increases with r that so does

$$R^{-2}b(R) = \int_0^1 (1 - t)B(Rt) dt.$$

Hence for $R > 2r$

$$\frac{b(R)}{(R - r)^2} \geq \frac{b(R)}{R^2} \geq \frac{b(2r)}{(2r)^2} = \frac{1}{4} \frac{b(2r)}{(2r - r)^2}.$$

Thus

$$\inf_{R \geq 2r} \frac{b(R)}{(R - r)^2} \geq \frac{1}{4} \frac{b(2r)}{(2r - r)^2}$$

and so

$$b_2(r) \geq \frac{1}{4} \min_{r < R \leq 2r} \frac{b(R)}{(R - r)^2}.$$

Thus R exists satisfying (4.11). Having chosen R to satisfy (4.11) we define h as in Lemma 4 and apply that lemma. We define p to be the smallest integer such that $p \geq 2$ and

$$2 \sin(\pi/2p) = |\exp(\pi i/p) - 1| < \frac{1}{2}h/r.$$

Then if $z_\nu = r \exp(2\pi i \nu/p)$, the disks $C_\nu: |z - z_\nu| < \frac{1}{2}h$, $\nu = 1, \dots, p$, cover $|z| = r$. Also

$$2\pi/p \geq \pi/(p-1) \geq 2 \sin(\pi/2(p-1)) \geq \frac{1}{2}h/r,$$

so that $p \leq 4\pi r/h$.

Again for $d < \frac{1}{2}h$ we have (4.9) in C_ν outside a set E_ν of disks the sum of whose radii is at most d . Since $d < \frac{1}{2}h < \frac{1}{2}r$ each exceptional disk $|z - z_j| < d_j \leq d$ meets $z = re^{i\theta}$ in an arc of diameter at most $2d_j$ and so length at most πd_j . Thus the total length of those arcs on $C_\nu \cap \{|z| = r\}$, which lie in the exceptional disks is at most πd . Thus (4.9) holds on $|\zeta| = r$, outside a set of arcs of total length at most $\pi p d$, i.e. (4.9) holds for $\zeta = re^{i\theta}$, $0 \leq \theta \leq 2\pi$, except for a set $e(r)$ of θ having measure

$$(4.12) \quad |e(r)| \leq \pi p d/r \leq (\pi d/r)(4\pi r/h) = 4\pi^2 d/h.$$

Further, for θ outside $e(r)$ we have from (4.4), (4.7) and (4.9)

$$(4.13) \quad \begin{aligned} u(re^{i\theta}) &= u(re^{i\theta}, \frac{1}{2}h) - N(re^{i\theta}, \frac{1}{2}h) \\ &> \frac{-b(R)}{(R-r)^2} \left(A_2 + A_4 \log \frac{16h}{d} \right) > -b_2(r) \left(4A_2 + 4A_4 \log \frac{16h}{d} \right) \end{aligned}$$

by (4.11). Suppose now that $K > 4A_2 + 4A_4 \log 32$. Then we define $d < \frac{1}{2}h$ by $K = 4A_2 + 4A_4 \log(16h/d)$ and deduce from (4.12) and (4.13) that (4.10) holds outside a set θ of measure

$$|e(r, K)| \leq 4\pi^2 d/h = 64\pi^2 \exp(A_2/A_4 - K/4A_4) \leq \exp(-K/8A_4)$$

if $K \geq A_6$. This proves Theorem 4 for $K \geq A_6$. Also, if $K < A_6$, (4.10) is trivial if $\exp(A_5 A_6) < 2$. Thus Theorem 4 holds in all cases with $A_5 = \inf(1/8A_4, (\log 2)/A_6)$.

We deduce the following consequence from Theorem 4, which may be considered as an analogue of the Edrei-Fuchs small arcs lemma [1, p. 322].

THEOREM 5. *If E is a set of measure $\delta < 2\pi$ on the interval $[0, 2\pi]$ then we have*

$$\int_E u(re^{i\theta}) d\theta > -A_7 b_2(r) \delta \log \left(\frac{4\pi}{\delta} \right).$$

We denote by $e(K)$ the set of θ such that $u(re^{i\theta}) < -Kb_2(r)$ and by $m(K)$ the measure of $e(K)$. Then Theorem 4 gives

$$\begin{aligned} \int_{e(K)} u(re^{i\theta}) d\theta &= b_2(r) \int_K^\infty t dm(t) = -b_2(r) \left\{ Km(K) + \int_K^\infty m(t) dt \right\} \\ &> -4\pi b_2(r) \left\{ K \exp(-A_5 K) + \int_K^\infty \exp(-A_5 t) dt \right\} \\ &> -A_9 b_2(r) \exp(-A_8 K). \end{aligned}$$

Given E as in Theorem 5 we choose $K > 0$, and define E_1, E_2 to be the subsets of E , where $u < -Kb_2(r)$, $u \geq -Kb_2(r)$, respectively. Then

$$\begin{aligned} \int_E u(re^{i\theta}) d\theta &= \int_{E_1} + \int_{E_2} \geq \int_{e(K)} u(re^{i\theta}) d\theta + \int_{E_2} u(re^{i\theta}) d\theta \\ &\geq -b_2(r) \{ A_9 \exp(-A_8 K) + K\delta \}. \end{aligned}$$

We choose K so that $A_9 \exp(-A_8 K) = \delta$, i.e. $K = (A_8)^{-1} \log(A_9/\delta)$, and deduce that

$$\int_E u(re^{i\theta}) d\theta > -b_2(r)\delta \left\{ 1 + \frac{1}{A_8} \log \frac{A_9}{\delta} \right\}$$

which gives Theorem 5.

We can now complete the proof of Theorem 1. Suppose that we have on the set E of values of r

$$(4.14) \quad T(r, f) > (1 - \varepsilon(r)^2) \log M(r, f),$$

where

$$(4.15) \quad \varepsilon(r) \rightarrow 0, \quad \text{but } \varepsilon(r)^2 \log M(r)/\log r \rightarrow \infty$$

as $r \rightarrow \infty$. We define $u(z) = \log |f(z)|$.

Let F be the set of all r , such that

$$(4.16) \quad b_2(r) > \frac{1}{\varepsilon(r)} B(r) = \frac{1}{\varepsilon(r)} b''(r), \quad \text{where } B(r) = \log M(r, f).$$

Then given $K > 1$, we have for all large r in F

$$b_2(r) \geq \frac{e^2 K^2}{4} b''(r),$$

so that F has upper density at most $2/(K+1)$ by Theorem 3. Since K is arbitrary, F has density zero.

Suppose now that a is any complex number and replace $u(z)$ by the function $u_a(z)$ defined by (4.6). Then

$$u_a(z) = \log |f(z) - a| + O(\log |z|)$$

so that for $|z| = r$

$$u_a^+(z) = \max(u_a(z), 0) = \{\log^+ |f(z)| + O(\log r)\}.$$

Thus, since $f(z)$ is transcendental we have $B(r, u_a(z)) = B(r) + O(\log r)$ as $r \rightarrow \infty$, and similarly $T(r, f_a(z)) = T(r, f) + O(\log r)$. Hence, also we have as $r \rightarrow \infty$

$$b(r, u_a) \sim b(r), \quad b_2(r, u_a) \sim b_2(r).$$

We deduce from (4.16) that for any complex a we have for $r \in E \setminus F$ and $r > r_0(a)$

$$(4.17) \quad b_2(r, u_a) < \frac{2}{\varepsilon(r)} B(r, u_a),$$

and from (4.14) and (4.15) that

$$(4.18) \quad T(r, f_a) > \{1 - 2\varepsilon(r)^2\} B(r, u_a).$$

Suppose now that for such a value of r , $e(r, a)$ is the set of all θ for which $u_a < 0$ and let $e'(r, a)$ be the complementary set of θ . Then

$$2\pi T(r, f_a) = \int_0^{2\pi} u_a^+(re^{i\theta}) d\theta = \int_{e'} + \int_{e'} \leq (2\pi - |e(r, a)|) B(r, u_a),$$

where $|e|$ denotes the measure of e . Thus

$$T(r, f_a) \leq \left(1 - \frac{|e(r, a)|}{2\pi}\right) B(r, u_a),$$

so that by (4.18), $|e(r, a)| \leq 4\pi\epsilon(r)^2$. Thus Theorem 5 and (4.17) yield for large r in $E \setminus F$

$$\begin{aligned} m(r, a) + O(\log r) &= \frac{-1}{2\pi} \int_{e(r, a)} u_a(re^{i\theta}) d\theta \\ &< A_7 b_2(r, u_a) |e(r, a)| \log \frac{4\pi}{|e(r, a)|} \\ &= O\left\{B(r, u_a) \epsilon(r) \log \frac{1}{\epsilon(r)}\right\} = o\{T(r, f)\}, \end{aligned}$$

and this proves Theorem 1, for $E \setminus F$ has positive upper density and so is unbounded.

5. Another growth lemma. In order to prove Theorem 2 we need

LEMMA 5. *Suppose that $B(r)$ is a positive increasing function of positive order, that $b(r)$ and $b_2(r)$ are defined by (4.2) and that $\varphi(r)$ is a positive function of r , such that*

$$(5.1) \quad \varphi(r) = O\{b_2(r)\} \quad \text{as } r \rightarrow \infty$$

and for some function $\epsilon(r)$, which decreases to zero as $r \rightarrow \infty$, we have

$$(5.2) \quad \varphi(r) = O\{\epsilon(r)b_2(r)\} \quad \text{as } r \rightarrow \infty$$

on a set E_1 of density one. Then there exists a set E_2 of upper density one, depending only on E_1 and the function $\epsilon(r)$, such that

$$(5.3) \quad \int_1^r \varphi(t) \log\left(\frac{r}{t}\right) \frac{dt}{t} = o\{B(r)\} \quad \text{as } r \rightarrow \infty$$

in E_2 .

We note that $b(r)$ and $b_2(r)$ also increase with r , and have positive order. In fact, the increasing property is obvious from (4.2) and

$$h^{-2}b(r+h) \geq h^{-2} \int_r^{r+h} (r+h-t)B(t) dt \geq \frac{1}{2}B(r)$$

so that $b_2(r) \geq \frac{1}{2}B(r)$ and $b(2r) \geq r^2B(r)/2$ for all r . Thus if $B(r)$ has positive order λ , $b(r)$ has order at least $\lambda + 2$ and $b_2(r)$ has order at least λ . We now choose μ such that $0 < \mu < \lambda$ and a sequence R_n , which tends to ∞ with n and is such that

$$(5.4) \quad b_2(r) \leq (r/R_n)^\mu b_2(R_n) \quad \text{for } 1 \leq r < R_n.$$

Since $b_2(r)/r^\mu$ is continuous and unbounded we may for instance choose $R_1 = 1$ and if R_{n-1} has been defined let R_n be the smallest number such that $R_n \geq 2R_{n-1}$ and

$$b_2(R_n)/R_n^\mu \geq \sup_{1 \leq R \leq 2R_{n-1}} b(r)/r^\mu.$$

We proceed to show that if K_n tends to ∞ sufficiently slowly with n and E_2 consists of all those points r in the intervals $[R_n, K_n R_n]$ for which

$$(5.5) \quad b_2(r) < K_n^{\mu/2} B(r),$$

then the set E_2 has the required property.

We note first that E_2 has upper density one. In fact, it follows from Theorem 3 that given $K > 1$ we have

$$(5.6) \quad b_2(r) < (eK/2)^2 B(r)$$

for a set of r in $[0, K_n R_n]$ having measure at least $(K-1)K_n R_n / (K+1) + O(1)$ when R_n is large and so in a set in $[R_n, K_n R_n]$ having measure at least

$$\left\{ \frac{K-1}{K+1} - \frac{1}{K_n} \right\} K_n R_n + O(1).$$

Thus, since (5.6) implies (5.5) for large n , we see that E_2 has upper density at least $(K-1)/(K+1)$, and since K can be as large as we please E_2 has upper density one.

We next choose the quantities K_n . Let E'_1 be the complement of E_1 , let $E'_1[r]$ be the intersection of E'_1 with the interval $[0, r]$, and let $|E'_1[r]|$ be the measure of $E'_1[r]$. Then we assume that K_n tends to infinity so slowly that

$$(5.7) \quad K_n^{2+\mu} < r/|E'_1(r)|, \quad r \geq R_n.$$

This is possible since E'_1 has density zero and $R_n \rightarrow \infty$ with n . We also assume that

$$(5.8) \quad K_n^\mu \epsilon(R_n/K_n) < 1,$$

which is possible since $\epsilon(r) \rightarrow 0$ as $r \rightarrow \infty$. The set E_2 defined as above is independent of $\varphi(r)$ and has upper density one. It remains to show that (5.3) holds in E_2 .

Assume that $r \in E_2$, $R_n \leq r \leq K_n R_n$, and write

$$(5.9) \quad I(r) = \int_1^r \varphi(t) \log \frac{r}{t} \frac{dt}{t} = I_0(r) + I_1(r) + I'_1(r),$$

where $I_0(r)$, $I_1(r)$ and $I'_1(r)$ are the integrals over the ranges $[1, R_n/K_n]$, $e_1 = [R_n/K_n, r] \cap E_1$ and $e'_1 = [R_n/K_n, r] \cap E'_1$, respectively. Then by (5.1), (5.4) and (5.5) we have

$$\begin{aligned} I_0(r) &= \int_1^{R_n/K_n} \varphi(t) \log \frac{r}{t} \frac{dt}{t} \leq 2 \int_1^{R_n/K_n} \varphi(t) \log \frac{R_n}{t} \frac{dt}{t} \\ &= O \left\{ \int_1^{R_n/K_n} b_2(t) \log \frac{R_n}{t} \frac{dt}{t} \right\} = O \left\{ b_2(R_n) \int_1^{R_n/K_n} \left(\frac{t}{R_n} \right)^\mu \log \frac{R_n}{t} \frac{dt}{t} \right\} \\ &= O \left\{ b_2(R_n) K_n^{-\mu} \log \frac{1}{K_n} \right\} = O \left\{ b_2(r) K_n^{-\mu} \log \frac{1}{K_n} \right\} \\ &= O \left\{ B(r) K_n^{-\mu/2} \log \frac{1}{K_n} \right\} = o \{ B(r) \}. \end{aligned}$$

Again by (5.2), (5.5) and (5.8)

$$\begin{aligned} I_1(r) &= O\left\{\varepsilon\left(\frac{R_n}{K_n}\right)\int_{e_1} b_2(t) \log \frac{r}{t} \frac{dt}{t}\right\} \\ &= O\left\{b_2(r)\varepsilon\left(\frac{R_n}{K_n}\right)\int_{r/K_n^2}^r \log \frac{r}{t} \frac{dt}{t}\right\} \\ &= O\left\{b_2(r)\varepsilon\left(\frac{R_n}{K_n}\right)(\log K_n)^2\right\} \\ &= O\left\{B(r)\varepsilon\left(\frac{R_n}{K_n}\right)K_n^{\mu/2}(\log K_n)^2\right\} = o\{B(r)\}. \end{aligned}$$

Finally, by (5.1), (5.5) and (5.7)

$$\begin{aligned} I'_1(r) &= O\left\{\int_{e'_1} b_2(t) \log \frac{r}{t} \frac{dt}{t}\right\} = O\left\{b_2(r)\left(\frac{K_n^2}{r}\right)\log K_n \int_{e'_1} dt\right\} \\ &= O\{B(r)K_n^{2+\mu/2}(\log K_n)r^{-1}|E'_1(r)|\} = o\{B(r)\}. \end{aligned}$$

Now (5.3) follows from (5.9) and Lemma 5 is proved.

6. Proof of Theorem 2. In order to prove Theorem 2 we need a formalism used elsewhere. We suppose that $f(z)$ is a transcendental entire function such that $f(0) = 1$ and denote by $n(r, \theta_1, \theta_2)$ the number of zeros of $f(z)$ in $0 < |z| < r$, $\theta_1 < \arg z < \theta_2$ each counted with due multiplicity. We also write

$$N(r, \theta_1, \theta_2) = \int_0^r n(t, \theta_1, \theta_2) \frac{dt}{t}.$$

Next, if $f(z) \neq 0$ on the segment $z = te^{i\theta}$, $0 \leq t \leq r$, we define $v(t, \theta)$ to be the continuous value of $\arg f(z)$ on this segment such that $v(0, \theta) = 0$, and we write,

$$V(r, \theta) = \frac{1}{2\pi} \int_0^r v(t, \theta) \frac{dt}{t}.$$

With this notation we have [3, Theorem 1]

LEMMA 6. If $f(z) \neq 0$ on the segments $z = te^{i\theta}$, $0 \leq t \leq r$, $\theta = \theta_1$ or θ_2 , then

$$N(r, \theta_1, \theta_2) = \frac{1}{2\pi} \int_{\theta_1}^{\theta_2} \log |f(re^{i\theta})| d\theta + V(r, \theta_1) - V(r, \theta_2).$$

We need to transform the quantity $V(r, \theta)$ a little and note that

$$v(r, \theta) = \int_0^r \frac{\partial v(t, \theta)}{\partial t} dt = \int_0^r -\frac{1}{t} \frac{\partial}{\partial \theta} \log |f(te^{i\theta})| dt.$$

Thus, for $\alpha < \beta \leq \alpha + 2\pi$ we have

$$\begin{aligned} \int_{\alpha}^{\beta} V(r, \theta) d\theta &= -\frac{1}{2\pi} \int_0^r \frac{ds}{s} \int_0^s \frac{dt}{t} \int_{\alpha}^{\beta} \log |f(te^{i\theta})| d\theta \\ &= \frac{1}{2\pi} \int_0^r \{\log |f(te^{i\alpha})| - \log |f(te^{i\beta})|\} \log \frac{r}{t} \frac{dt}{t}. \end{aligned}$$

We write

$$(6.1) \quad M(t) = \sup_{0 \leq \theta \leq 2\pi} |f(te^{i\theta})|, \quad B(t) = \log M(t)$$

and

$$\varphi(t, \alpha) = \frac{1}{2\pi} \log \frac{M(t)}{|f(te^{i\alpha})|}.$$

Thus

$$(6.2) \quad \int_{\alpha}^{\beta} V(r, \theta) d\theta = \int_0^r \{ \varphi(t, \beta) - \varphi(t, \alpha) \} \log \frac{r}{t} \frac{dt}{t}.$$

Our aim is to show that the positive function $\varphi(t, \alpha)$ is on the average not too large. We also define $\varphi_a(t, \alpha)$ to be the function $\varphi(t, \alpha)$ defined as above w.r.t. the functions $f_a(z)$ introduced in (4.6).

LEMMA 7. *If*

$$\varphi_a(t) = \frac{1}{2\pi} \int_0^{2\pi} \varphi_a(t, \alpha) d\alpha,$$

then under the hypotheses of Theorem 2 there exists a set E_2 of upper density one such that we have

$$\int_1^r \varphi_a(t) \log \frac{r}{t} \frac{dt}{t} = o\{B(r)\}$$

as $r \rightarrow \infty$ in E_2 for each complex a .

We define the set E_2 as in Lemma 5, where $B(r)$ is given by (6.1) (i.e. for the function $f_0(z) = f(z)$). Then $B(r)$ has positive order by hypothesis, so that Lemma 5 is applicable. We deduce from Theorem 1 that under the hypotheses of Theorem 1, we have for each complex a as $r \rightarrow \infty$ in E_1

$$(6.3) \quad \begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f_a(re^{i\theta})| d\theta &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| + O(\log r) \\ &> B(r) - \varepsilon(r)B(r) + O(\log r), \end{aligned}$$

where $\varepsilon(r) \rightarrow 0$ with r and $\varepsilon(r)$ is independent of a . Thus $|f_a(re^{i\theta})| > 1$ outside a set of θ of measure at most $O\{\varepsilon(r)\}$, provided that $\varepsilon(r) \rightarrow 0$ so slowly that $\log r = o\{\varepsilon(r)B(r)\}$ as $r \rightarrow \infty$. Now Theorem 5 shows that as $r \rightarrow \infty$ in E_1 we have

$$(6.4) \quad \begin{aligned} \int_0^{2\pi} \log^+ \left| \frac{1}{f_a(re^{i\theta})} \right| d\theta &= O\left\{ \varepsilon(r) \log \frac{1}{\varepsilon(r)} b_2(r) \right\} \\ &= O\{ \varepsilon_1(r) b_2(r) \}, \end{aligned}$$

where $\varepsilon_1(r)$ is independent of a . We note that if $a \neq 1$, $f_a(z) = (f(z) - a)/(1 - a)$ while $f_1(z) = (f(z) - 1)/z^\lambda$. Thus for any fixed a and large r we have

$$M_a(r) = \sup_{|z|=r} |f_a(z)| < C_a M(r), \quad |z| = r,$$

where the constant C_a depends only on a . Also (6.3) and (6.4) yield

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \log \frac{M_a(r)}{|f_a(re^{i\theta})|} d\theta &= O\{\varepsilon(r)B(r) + \varepsilon_1(r)b_2(r) + \log r\} \\ &= O\{\varepsilon_1(r)b_2(r)\} \end{aligned}$$

as $r \rightarrow \infty$ in E_1 . Thus, if $\varphi_a(t)$ is defined with $f_a(z)$ instead of $f(z)$, we see that for each complex a

$$\varphi_a(t) = \frac{1}{2\pi} \int_0^{2\pi} \varphi_a(t, \alpha) dt = O\{\varepsilon_1(t)b_2(t)\}$$

as $t \rightarrow \infty$ in E_1 , while we have in any case

$$\varphi_a(t) = B(t) - \frac{1}{2\pi} \int_0^{2\pi} \log |f_a(te^{i\theta})| d\theta + O(1) \leq B(t) + O(1) = O\{b_2(t)\}.$$

Thus $\varphi_a(t)$ satisfies the hypotheses of Lemma 5 and we deduce that there exists a set E_2 of upper density one such that for each complex a we have

$$(6.5) \quad \int_1^r \varphi_a(t) \log \frac{r}{t} \frac{dt}{t} = o\{B(r)\} \quad \text{as } r \rightarrow \infty \text{ on } E_2.$$

This proves Lemma 7.

We now suppose that we are given $\varepsilon > 0$ and θ_1, θ_2 , such that $\theta_1 \leq \theta_2 < \theta_1 + 2\pi$. We also take a fixed complex a and assume that $r \rightarrow \infty$ on E_2 . Then there exist α, β such that $\theta_2 < \alpha < \theta_2 + \varepsilon/3$, $\theta_2 + 2\varepsilon/3 < \beta < \theta_2 + \varepsilon$ and

$$\begin{aligned} 0 &\leq \int_1^r \varphi_a(t, \alpha) \log \left(\frac{r}{t} \right) \frac{dt}{t} \leq \frac{3}{\varepsilon} \int_1^r \log \frac{r}{t} \frac{dt}{t} \int_{\theta_2}^{\theta_2 + \varepsilon/3} \varphi_a(t, \theta) d\theta \\ &\leq \frac{6\pi}{\varepsilon} \int_1^r \log \frac{r}{t} \varphi_a(t) \frac{dt}{t}, \\ 0 &\leq \int_1^r \varphi_a(t, \beta) \log \frac{r}{t} \frac{dt}{t} \leq \frac{6\pi}{\varepsilon} \int_1^r \log \varphi_a(t) \frac{dt}{t}. \end{aligned}$$

Using (6.2) and (6.5) we deduce that $|\int_{\alpha}^{\beta} V(r, \theta) d\theta| = o\{B(r)\}$. Hence there exists $\varphi_2 = \varphi_2(r)$ such that $\alpha < \varphi_2 < \beta$ and so $\theta_2 < \varphi_2 < \theta_2 + \varepsilon$ and $V(r, \varphi_2) = o\{B(r)\}$. Similarly, there exists φ_1 , such that $\theta_1 - \varepsilon < \varphi_1 < \theta_1$, and $V(r, \varphi_1) = o\{B(r)\}$. Thus Lemma 6 shows that

$$\begin{aligned} N(r, \theta_1, \theta_2) &\leq N(r, \varphi_1, \varphi_2) \leq \frac{\varphi_2 - \varphi_1}{2\pi} \{B(r) + O(1)\} + o\{B(r)\} \\ &\leq \frac{\theta_2 - \theta_1 + 2\varepsilon + o(1)}{2\pi} B(r). \end{aligned}$$

This gives

$$(6.6) \quad \overline{\lim} \frac{N(r, \theta_1, \theta_2)}{B(r)} \leq \frac{\theta_2 - \theta_1}{2\pi}.$$

Also, we may assume that E_2 is disjoint from the set F of Theorem 1, since this does not affect the density. Then as $r \rightarrow \infty$ in E_2

$$(6.7) \quad N(r, \theta_2, \theta_1 + 2\pi) + N(r, \theta_1, \theta_2) = N(r) = (1 + o(1))B(r)$$

by Theorem 1. We apply (6.6) with $\theta_2, \theta_1 + 2\pi$ instead of θ_1, θ_2 and obtain

$$\varlimsup \frac{N(r, \theta_2, \theta_1 + 2\pi)}{B(r)} \leq \frac{2\pi + \theta_1 - \theta_2}{2\pi}.$$

Now (6.7) gives

$$\varliminf \frac{N(r, \theta_1, \theta_2)}{B(r)} \geq \frac{\theta_2 - \theta_1}{2\pi}.$$

Combining this with (6.6) we obtain

$$\lim \frac{N(r, \theta_1, \theta_2)}{B(r)} = \frac{\theta_2 - \theta_1}{2\pi}$$

as $r \rightarrow \infty$ in E_2 , and this proves Theorem 2.

In conclusion we note that, by Theorem 2, (1.2) implies angular equidistribution of all a -values unless $f(z)$ has order zero. However, for functions of order zero it follows from Theorem 3 of [3] that $f(z)$ satisfies the conclusion of Theorem 2 if (1.4) holds and a fortiori if (1.1) holds. Thus (1.1) always implies equidistribution of the a -values.

REFERENCES

1. A Edrei and W. H. J. Fuchs, *Bounds for the number of deficient values of certain classes of meromorphic functions*, Proc. London Math. Soc. (3) **12** (1962), 315–344.
2. W. K. Hayman, *The minimum modulus of large integral functions*, Proc. London Math. Soc. (3) **2** (1952), 469–512.
3. ———, *Angular value distribution of power series with gaps*, Proc. London Math. Soc. (3) **24** (1972), 590–624.
4. ———, *On Iversen's theorem for meromorphic functions with few poles*, Acta Math. **141** (1978), 115–145.
5. W. K. Hayman and F. M. Stewart, *Real inequalities with applications to function theory*, Proc. Cambridge Philos. Soc. **50** (1954), 250–260.
6. T. Murai, *The deficiency of entire functions with Fejér gaps*, Ann. Inst. Fourier (Grenoble) **33** (1983), 39–58.

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